THE EXOTIC STATISTICS OF LEAPFROGGING SMOKE RINGS

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Abstract

The leapfrogging motion of smoke rings is a three dimensional version of the motion that in two dimensions leads to exotic exchange statistics. The statistical phase factor can be computed using the hydrodynamical Euler equation, which is a universal law for describing the properties of a large class of fluids. This suggests that three dimensional exotic exchange statistics is a common property of closed vortex loops in a variety of quantum liquids and gases, from helium superfluids to Bose-Einstein condensed alkali gases, metallic hydrogen in its liquid phases and maybe even nuclear matter in extreme conditions.

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Closed vortex loops such as an ordinary smoke ring, are common in inviscid and low viscosity fluids. They are among the simplest examples of organized structure formation, and increasingly important in a variety of scenarios. In cosmology, the theory of cosmic string loops might explain a myriad of phenomena from baryon number asymmetry to galactic structure formation [1]. In strong interaction physics the string that confines quarks might also exist in solitude as a closed string that describes glueballs [2], and the physical principles that govern the behaviour of the confining string can also explain the origin of most of the mass observed in the Universe [3]. In condensed matter physics, the properties of closed vortex loops are under an intense scrutiny in a variety of fluids, including quantum superfluids [4], atomic Bose-Einstein condensed alkali gases [5], nematic liquid crystals [6], and the liquid phases of metallic hydrogen [7]. Finally, the dynamics of vortex loops is highly relevant to hydrodynamical turbulence [8], which is widely considered as *the* unexplained phenomenon in classical physics [3].

The interactions between vortex loops can lead to stunning phenomena. One of the most impressive sights occurs when two identical smoke rings leapfrog through each other as shown in Figure 1.
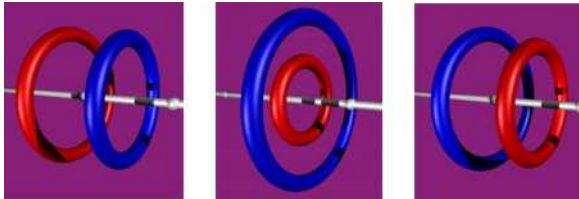

FIG. 1: The leapfrogging motion of two smoke rings along their common axis begins with the rear (red) ring contracting and accelerating, at the same time the front (blue) ring slows down and expands its diameter (left picture). Eventually the rear ring catches up with the front ring, slides through its center (center picture), and emerges ahead of it (right picture). This reverses the role of the smoke rings and under ideal conditions the process repeats itself almost perpetually.

In the present letter we propose that the leapfrogging motion is the three dimensional version of the motion that in a two dimensional context leads to exotic exchange statistics [9]:

In certain two dimensional physical systems such as very thin (^{4}He) superfluid films, fractional quantum Hall systems, and high temperature superconductors, identical point

particles which are subject to the laws of quantum mechanics do not need to obey the conventional bosonic and fermionic exchange statistics. Instead, under exchange their quantum mechanical wave function acquires a phase factor $(-1)^\Phi$, where the statistical phase Φ can have *any* value [9]. In principle, this phase can be made visible by interference, when a pointlike particle such as a point vortex in a thin ^{4}He superfluid film traverses around a closed loop Γ. The quantum mechanical phase Φ displays the following two (universal) contributions [10], [9]

$$\Phi = 2\pi Q(\rho_0 A - n \cdot \pi \xi^2 \rho_V) \tag{1}$$

Here A is the area encircled by Γ, Q is the circulation of the vortex which is being traversed as a multiplet of h/m with m the mass of the background fluid particles, and ρ_0 is the constant London limit density of the background fluid. The integer n counts the number of vortices with circulation Q that are being encircled by Γ. The parameter ξ is the healing length that characterizes the thickness of the vortex core, and ρ_V is an averaged deficit in the density of background fluid particles inside the core.

The first term in (1) is a Bohm-Aharonov phase factor for transporting an isolated pointlike vortex around Γ. It is insensitive to the presence of additional vortices and does not contribute to any exotic statistics. The second term in (1) relates directly to the presence of vortices that reside in the area A, and this is the contribution that leads to exotic statistics.

The result (1) idealizes a vortex as a circular object with an effective core radius ξ which acts as a short distance cut off scale beyond which a model dependent description of the vortex structure becomes relevant. The *universal* feature of (1) is that independently of model dependent details, the exotic contribution to statistics measures how the presence of vortices influences the number of background fluid particles within the area that is being encircled by Γ.

We shall now proceed to derive an analogous universal law in the three dimensional context:

In a three dimensional physical world the exchange of pointlike particles is topologically trivial. The pertinent quantum mechanical wave function can only exhibit either the Bose-Einstein ($\Phi = 0$) or the Fermi-Dirac ($\Phi = 1$) statistics, even though this point of view has been occasionally challenged [11]. But if instead of pointlike particles we consider the exchange of closed three dimensional loops and allow for motions such as the leapfrogging motion of two smoke rings, the exchange topology becomes nontrivial. In fact, the motion

group of leapfrogging loops contains Artin's braid group [12] which in the two dimensional context leads to a calculable exotic exchange statistics. This suggests that closed three dimensional loops should also exhibit exotic statistical behaviour. In fact, in [13] a scenario has been proposed for computing the ensuing statistical phase in the case of closed strings. But the approach in [13] assumes that the strings carry a nontrivial internal structure. As a consequence the result is a reflection of phenomena which reside in different dimensionalities, and in a hydrodynamical context the approach would lead to a conflict with Kelvin's circulation theorem.

Here we show that in fluids, the exchange of structureless closed vortex loops leads to computable exotic exchange statistics. The ensuing statistical phase can be evaluated directly from the hydrodynamical Euler equation which is widely considered as *the* universal equation for describing homogeneous and inviscid liquids and gases above microscopic distance scales. As a consequence our result suggests that three dimensional exotic exchange statistics is commonplace, it can be present whenever closed vortex loops are exchanged.

The Euler equation reads [8]

$$\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{F} \tag{2}$$

where the fluid velocity $\mathbf{u}$ and the fluid density ρ are subject to the continuity equation

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3}$$

For us the details of the (conservative) external force $\mathbf{F}$ in (2) are quite inessential. For simplicity we assume that the flow is isentropic so that any external force can be combined with the pressure p into an enthalpy $V(\rho)$ which only depends on the density ρ. If $V(\rho) \sim \lambda(\rho^2 - \rho_0^2)^2$ with λ large, the fluid density is then practically a constant $\rho(\mathbf{x}) \approx \rho_0$ and this defines the London limit.

The version of (2) and (3) which is relevant for describing the fluid above microscopic scales derives from the classical action

$$S = \int d^3x dt \left\{ \rho\dot{\theta} - \theta\nabla \cdot (\rho\mathbf{u}) - \frac{1}{2}\rho\mathbf{u}^2 - V[\rho] \right\} \tag{4}$$

Indeed, a variation of θ in (4) yields (3), a variation of $\mathbf{u}$ gives

$$\mathbf{u} = \nabla\theta \quad \text{when} \quad \rho(\mathbf{x}) \neq 0 \tag{5}$$

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and a variation of ρ leads to the Euler equation (2) when we use (5). Note that the action (4) has the standard Hamiltonian form with θ and ρ as canonical conjugates. Consequently it allows, in principle, for a quantum mechanical inspection of the exchange statistics of vortex loops in fluids which are subject to quantum laws.

Instead of the (universal) Euler equation we could certainly employ some more sophisticated model. In a refined analysis this may even become practical since among its limitations, (4) yields a gapless energy spectrum. But since the effect we consider has a long distance component that would not lead to any modification in our main conclusions: The Euler equation is a universal model for describing the behaviour of more elaborate quantum fluids at large length scales, and the very fact that it encodes exotic statistics is a clear indication that our observations are similarly universal and have a wide applicability.

A closed vortex loop defines a region in the fluid where vorticity is nonvanishing, $\nabla \times \mathbf{u} \neq 0$. Since (5) states that vorticity vanishes whenever $\rho(\mathbf{x}) \neq 0$, for consistency the fluid density $\rho(\mathbf{x})$ must vanish at the core of the vortex loop. For simplicity we imagine the vortex loop as a very thin, but not necessarily very short tubular region in which the vorticity has its support and the fluid density $\rho(\mathbf{x}) \approx 0$. The thickness ξ of the tube defines a short distance cut off scale beyond which (4) lacks validity. In the presence of closed vortex loops the region where $\rho \neq 0$ is then multiply connected, the circulation of $\mathbf{u}$ along irreducible curves that encircle the vortex loops does not need to vanish, and the variable θ becomes in general multivalued.

We realize the present scenario by describing a thin vortex tube as a closed curve $\mathbf{R}(s)$ where s measures distance along the curve. Our consistency requirement for the fluid density states that on these curves $\rho[\mathbf{R}(s)] = 0$ while outside of the vortices the density very rapidly (over a distance $\approx \xi$) approaches its constant London limit value $\rho(\mathbf{x}) \approx \rho_0$.

In the vicinity of a vortex loop the velocity $\mathbf{u}(\mathbf{x})$ is computed from the Biot-Savart law [8]

$$\mathbf{u}(\mathbf{x}) = Q \cdot \nabla \times \oint \frac{d\mathbf{s}}{|\mathbf{x} - \mathbf{R}(s)|} + \nabla \psi \tag{6}$$

where Q is (4π times) the circulation of the vortex loop, and ψ is some (single valued) function in the fluid that we include for completeness as the fluid does not need to be incompressible. The vorticity has indeed support only on the curve $\mathbf{R}(s)$

$$\nabla \times \mathbf{u} = Q \cdot \oint d\mathbf{R} \, \delta(\mathbf{x} - \mathbf{R}) \tag{7}$$

and outside of the curve we have indeed a potential flow, with velocity field $\mathbf{u}$ given by

$$\mathbf{u}(\mathbf{x}) \;=\; -Q \cdot \nabla\Omega(\mathbf{x}) + \nabla\psi \tag{8}$$

where $\Omega(\mathbf{x})$ is the (signed) solid angle at the point $\mathbf{x}$ which is subtended by some two dimensional surface Σ which is bounded by $\mathbf{R}(s)$. The solid angle has a number of valuable properties which are obtained by inspecting the two-form [14]

$$\hat{\omega}(\mathbf{x}_0) \;=\; \frac{1}{2}\epsilon_{ijk}\frac{(x^i - x_0^i)dx^j \wedge dx^k}{|\mathbf{x} - \mathbf{x}_0|^3} \tag{9}$$

Its integral

$$\Omega(\mathbf{x}_0) = \int\limits_{\Sigma} d^2\sigma\, \epsilon^{ab}\epsilon_{ijk}\frac{\partial z^i}{\partial \sigma^a}\frac{\partial z^j}{\partial \sigma^b}\frac{x^k - z^k}{|\mathbf{x} - \mathbf{z}|^3} \tag{10}$$

over the surface Σ coincides with the (signed) solid angle which is subtended by Σ at $\mathbf{x}_0$. Since $d\hat{\omega} = 0$ the solid angle remains intact under such local deformations of the surface Σ that leave the boundary curve $\mathbf{R}(s)$ intact: The solid angle depends only on the boundary curve, and when we move around the boundary curve by linking it once the solid angle jumps by $\pm 4\pi$. Consequently in the presence of closed vortex loops the variable θ in (5) indeed becomes multivalued.

We now consider an (imaginary) periodic adiabatic motion of a closed vortex loop $\mathbf{R}(s)$ in the fluid. We parametrize the motion by $\mathbf{R}(s,t)$ where t is the adiabatic time, and since the motion is periodic we have $\mathbf{R}(s,0) = \mathbf{R}(s,T)$ for some T. The surface $\mathbf{R}(s,t)$ encloses a toroidal volume V_T in the fluid. We first assume that V_T does not intersect or enclose any other vortices. In the London limit we then get from (4), (5), (8) for the quantum mechanical adiabatic phase [9]

$$\Phi = \int d^3x dt\, \rho\dot{\theta}$$

$$= -\lim_{\Delta t \to 0} Q \cdot \int d^3x \int\limits_0^T dt\, \rho(\mathbf{x},t)\frac{\Omega(\mathbf{x}, t+\Delta t) - \Omega(\mathbf{x}, t)}{\Delta t} \tag{11}$$

The difference $\Omega(t + \Delta t) - \Omega(t)$ in (11) emanates an evolution of the surfaces $\Sigma(t)$ which is characterized by an evolution of the coordinates $z^i(\sigma^a; t)$ that embed $\Sigma(t)$ in the fluid. The general evolution of the z^i admits two contributions, there is a flow which is normal to the surface $\Sigma(t)$ and there is a flow which is tangential to this surface. The latter corresponds to a reparametrization of $\Sigma(t)$ which we exclude. When we substitute the evolution of z^i

normal to the surface in (11) and assume the London limit of constant density we get for the quantum mechanical phase

$$\Phi \;=\; \frac{4\pi}{3}\rho_0 Q \int\limits_{V_T} d^3\sigma \; \epsilon^{\alpha\beta\gamma}\epsilon_{ijk}\frac{\partial z^i}{\partial\sigma^\alpha}\frac{\partial z^j}{\partial\sigma^\beta}\frac{\partial z^k}{\partial\sigma^\gamma} \;=\; \frac{4\pi}{3}Q\rho_0 V_T \tag{12}$$

where $\sigma^3 = t$ and we have used the fact that the integrand coincides with the volume three-form. The result should be compared to the first term in (1): Clearly, (12) is a three dimensional version of that term.

We now proceed to the general case where the periodic adiabatic transport $\mathbf{R}(s,t)$ of the closed vortex loop surrounds, without touching, a number of other closed vortex loops in the fluid. This could for example relate to the leapfrogging motion of two identical vortex loops. The integral (11) now acquires two distinct contributions. One of these contributions is an integral that extends over that subregion of the volume V_T which does not contain any vortices. In this subregion we are at the London limit with $\rho(\mathbf{x}) = \rho_0$ a constant, and the ensuing integral in (12) leads to a contribution where V_T becomes replaced by the volume of the region where there are no vortices. The second contribution is an integral over the volume occupied by the vortices. In this case, we recall that the consistency of our equations implies that at the location of vortices we have $\rho(\mathbf{x}) = 0$. Consequently the ensuing contribution to the integral (11) vanishes. Combining the two contributions we conclude that in the presence of N vortices with circulation $n_i Q$, length L_i and (average) tube radius ξ_i we get for the quantum mechanical phase

$$\Phi \;=\; \frac{4\pi}{3}\rho_0 Q(V_T - \pi \sum_i^N n_i L_i \xi_i^2) \tag{13}$$

where the second term adds up to the total volume of the vortices that are being encircled; the present model does not allow for a more detailed computation of this additional contribution as it involves Physics at length scales which is beyond the reach of a simple Euler equation.

The final result (13) is clearly a direct three dimensional generalization of (1). In particular, when two identical vortex loops leapfrog through each other, in addition to the conventional Bohm-Aharonov contribution to the quantum mechanical phase we also have a contribution which is proportional to the (average) volume of the vortices under the leapfrogging period. It is this additional contribution that leads to an exotic exchange statistics, in full parallel to the two dimensional case.

The result (13) is phenomenological in the sense that it approximates a vortex as a circular tube with radius ξ which supports the vorticity. But we argue that if we repeat our computation in a more involved model we arrive at the same universal structure. For this, consider *e.g.* a complex scalar field ψ which relates to the order parameter of various quantum fluids. If we write $\psi = \sqrt{\rho}\exp(i\theta)$ the ensuing kinetic term acquires the same functional form with the kinetic term in (4),

$$\int d^3x dt \frac{i}{2}\{\psi^*\dot{\psi} - \dot{\psi}^*\psi\} = \int d^3x dt \, \rho\dot{\theta}$$

If we repeat the present computation, the only difference will then be that instead of a step-like density profile $\rho(\mathbf{x})$ that vanishes inside the vortex tube and acquires its nonvanishing asymptotic London limit outside of the tube, we now have a density profile that vanishes at the center line of the vortex and approaches the London limit at an exponential rate in the healing length ξ. Since this difference does not introduce any qualitative changes, we conclude that (13) is quite universal.

In conclusion, we have shown that whenever the large scale motion of closed vortex loops in a quantum fluid can be governed by the hydrodynamical Euler equation the ensuing statistical phase acquires an exotic contribution, which is fully analogous to the contribution that in two dimensions leads to anyon statistics. The simplicity of our derivation and the universality of the Euler equation in describing the long distance properties of (non-relativistic) quantum fluids is an indication that three dimensional exotic statistics is generic. Since vorticity is pivotal to the properties of low viscosity and inviscid liquids and gases we expect that our observations will have wide applicability to the thermodynamics of fluids, from the onset of turbulence to dynamics of phase transitions.

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